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Higher order asymptotics of Toeplitz determinants with symbols in weighted Wiener algebras ☆

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Abstract

We extend a result of Böttcher and Silbermann on higher order asymptotics of determinants of block Toeplitz matrices with symbols in Wiener algebras with power weights to the case of Wiener algebras with general weights satisfying natural submultiplicativity, monotonicity, and regularity conditions.

Keywords: Toeplitz determinant; Weighted Wiener algebra; Canonical Wiener–Hopf factorization; Strong Szegő–Widom limit theorem; Schatten–von Neumann class; p-Regularized operator determinant

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_-$, $\mathbb{Z}_+$, and $\mathbb{C}$ be the sets of integers, positive integers, negative integers, non-negative integers, and all complex number, respectively. Suppose $N \in \mathbb{N}$. A *block Toeplitz matrix* is a matrix of the form

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{1}$$

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where $\{a_k\}_{k\in\mathbb{Z}}$ is a sequence of $N \times N$ matrices. For a Banach space X, let X_N and $X_{N\times N}$ be the spaces of vectors and matrices with entries in X. We will consider the norm $\|(x_1,\dots,x_N)\|_{X_N} = (\|x_1\|_X^2 + \cdots + \|x_N\|_X^2)^{1/2}$. Let $\mathbb{T}$ be the unit circle, $L^\infty := L^\infty(\mathbb{T})$ and $H^2 := H^2(\mathbb{T})$ be the standard Hardy space of the unit circle. It is well known that the matrix (1) induces a bounded (Toeplitz) operator $T(a)$ on H_N^2 if and only if there exists a matrix function $a \in L^\infty_{N\times N}$ such that

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a\big(e^{i\theta}\big)e^{-ik\theta}\,d\theta \quad (k\in\mathbb{Z})$$

is the sequence of Fourier coefficients of a. The function a is called the *symbol* of $T(a)$. Let $T_n(a) := [a_{j-k}]_{j,k=0}^n$ $(n \in \mathbb{Z}_+)$ be the truncated Toeplitz matrix. We are interested in the asymptotic behavior of the determinants of the truncated block Toeplitz matrices (Toeplitz determinants) $D_n(a) := \det T_n(a)$ as $n \to \infty$.

The strong Szegő limit theorem (in the scalar case $N = 1$) states that under certain conditions one has for the Toeplitz determinants the asymptotic formula

$$D_n(a) \sim G(a)^{n+1}E(a), \tag{2}$$

with completely identified constants $G(a)$ and $E(a)$. Sufficient conditions for the validity of (2) have been given by many authors. Widom [22] extended these results to the block case ($N > 1$) and proved that if $a \in L^\infty_{N\times N}$ satisfies

$$\sum_{k=-\infty}^{\infty} \|a_k\|^2\big(|k|+1\big) < \infty, \tag{3}$$

where $\|\cdot\|$ is any matrix norm on $\mathbb{C}_{N\times N}$, and $T(a)$ is a Fredholm operator on H_N^2 of index zero, then (2) is fulfilled with

$$G(a) := \lim_{r\to 1-0} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\det a_r\big(e^{i\theta}\big)\,d\theta\right), \quad a_r\big(e^{i\theta}\big) := \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}, \tag{4}$$

and $E(a) := \det T(a)T(a^{-1})$, where the last det refers to the determinant defined for operators on Hilbert space differing from the identity by an operator of trace class [17, Chapter 4]. Böttcher and Silbermann [5] proved the formula (2) in the block case $N > 1$ for various classes of symbols. We refer to [6, Chapter 6], [7, Chapter 10], [8, Chapter 5] for the history, exact references, and proofs. We mention here also recent papers [9,10,18], where alternative approaches to the proof of the strong Szegő–Widom limit theorem are suggested, and the author's paper with Santos [19], where the strong Sezgő limit theorem is proved in the scalar case for a new class of symbols with nonstandard smoothness.

Note that the perhaps simplest proof of the strong Szegő–Widom limit theorem is based on the so-called Borodin–Okounkov identity (this is a commonly used name, although it is not entirely historically correct)

$$\det T_n(a) = G(a)^{n+1}E(a)\det\big(I - Q_nH(b)H(\tilde{c})Q_n\big) \quad \text{for all } n\in\mathbb{N},$$

where the projections Q_n are defined in Section 4.1. One can show that if (3) is fulfilled and $T(a)$, $T(\tilde{a})$, where $\tilde{a}(t) := a(1/t)$ for $t \in \mathbb{T}$, are invertible on H^2_N, then the product of Hankel operators $H(b)H(\tilde{c})$ is correctly defined as in Section 2.2 and it belongs to the trace class. This beautiful identity has a complicated history. As far as we know, it was first obtained and used to prove the Szegő strong limit theorem by Geronimo and Case [13] in 1979. Then it was rediscovered (in the scalar case) by Borodin and Okounkov twenty years later [2], who answered a question posed by Deift and Its. This identity was extended to the block case by Basor and Widom [1], who found three proofs of it. Several simpler proofs were also found by Böttcher [3] (see also references given there). For a further development of this topic we recommend to consult a recent monograph by Simon [21]. However, the problems remain when we cannot guarantee that the product of Hankel operators $H(b)H(\tilde{c})$ is of trace class. We will touch the situation when $H(b)H(\tilde{c})$ belongs only to the Schatten–von Neumann class $\mathcal{C}_p(H^2_N)$ for $p \in \mathbb{N} \setminus \{1\}$.

Fisher and Hartwig [11] were probably the first to draw due attention to higher order correction terms in Szegő's asymptotic formula (2). Böttcher and Silbermann (see [5], [6, 6.15–6.23], [7, 10.32–10.36]) gave higher order asymptotic terms in the asymptotic formulas for Toeplitz determinants with symbols in weighted Wiener algebras

$$(W_\omega)_{N\times N} := \left\{ a : \mathbb{T} \to \mathbb{C}_{N\times N} \colon a(t) = \sum_{n=-\infty}^{\infty} a_n t^n,\ \sum_{n=-\infty}^{\infty} \|a_n\| \omega(n) < \infty \right\}$$

with the power weight

$$\omega(n) = \varrho(n) := \begin{cases} (-n+1)^\alpha & \text{for } n \in \mathbb{N}_-, \\ (n+1)^\beta & \text{for } n \in \mathbb{Z}_+ \end{cases} \qquad (\alpha, \beta > 0). \tag{5}$$

They considered also many other classes of symbols, for instance, the algebras of Hölder continuous functions.

The aim of this paper is to extend Böttcher and Silbermann's higher order asymptotic formulas for Toeplitz determinants (see Theorem 21) to the case of symbols that belong to more general weighted Wiener algebras $(W_\omega)_{N\times N}$, where the weight $\omega : \mathbb{Z} \to [1, \infty)$ satisfies

$$1 \leqslant \omega(i+j) \leqslant \omega(i)\omega(j) \quad (i, j \in \mathbb{Z}), \tag{6}$$

$$\omega(\pm n) \leqslant \omega\big(\pm(n+1)\big) \quad (n \in \mathbb{Z}_+), \tag{7}$$

$$\lim_{n\to+\infty} \sqrt[n]{\omega(n)} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\omega(-n)}} = 1. \tag{8}$$

The paper is organized as follows. In Section 2 we study canonical right and left Wiener–Hopf factorizations in weighted Wiener algebras $(W_\omega)_{N\times N}$. In Section 3 we collect necessary facts about p-regularized determinants of operators $I + K$, where K belongs to the Schatten–von Neumann classes $\mathcal{C}_p(\mathcal{H})$ and $p \in \mathbb{N}$. Section 4 contains a preliminary asymptotic analysis of Toeplitz determinants based on right and left canonical Wiener–Hopf factorizations of the symbol a. We write down explicitly asymptotic formulas for Toeplitz determinants $T_n(a)$ if a admits canonical right and left Wiener–Hopf factorizations and the operators $H(b)H(\tilde{c})$ and $H(\tilde{c})H(b)$ belong to $\mathcal{C}_p(H^2_N)$. This approach goes back to Böttcher and Silbermann [5], [6, 6.15–6.23], [7, Theorem 10.35]. Theorem 15 is contained there implicitly. However, we wish to give a self-contained proof. In Section 5 we apply Theorem 15 to weighted Wiener algebras with weights satisfying (6)–(8) and the following two regularity conditions. If

$$\sum_{k=1}^{\infty}\big(\omega(k)\omega(-k)\big)^{-p}<\infty, \tag{9}$$

then $H(b)H(\tilde{c})$ and $H(\tilde{c})H(b)$ belong to $\mathcal{C}_p(H^2_N)$. If, in addition,

$$\lim_{n\to\infty}\big(\omega(n)\omega(-n)\big)^{1-p}\sum_{j=1}^{n}\big(\omega(j)\omega(-j)\big)^{-1}=0, \tag{10}$$

then we can simplify a little bit the asymptotic formulas (see Theorem 20(c)), removing the correcting term $F_{n,p-1}$. For the power weight (5), our conditions (9) and (10) hold if $\alpha+\beta>1/p$ (see Theorem 21). Finally, we show that our Theorem 20 is stronger than Böttcher and Silbermann's Theorem 21, constructing an example of a weight ω and a function $a\in W_\omega$ (see Theorem 23) such that Theorem 20 is applicable to a, but Theorem 21 is not.

2. Wiener–Hopf factorizations in weighted Wiener algebras

2.1. Weighted Wiener algebras and their maximal ideal spaces

It is well known that if (6) is fulfilled, then $(W_\omega)_{N\times N}$ is a Banach algebra with respect to the norm

$$\|a\|_{\omega,N}:=\sum_{k=-\infty}^{\infty}\|a_k\|\omega(k)$$

and $(W_\omega)_{N\times N}\subset W_{N\times N}\subset C_{N\times N}$. Here W is the standard Wiener algebra of scalar functions with absolutely convergent Fourier series and $C=C(\mathbb{T})$ is the set of all continuous scalar functions on $\mathbb{T}$. Condition (6) implies that the limits

$$R_-(\omega):=\lim_{n\to+\infty}\frac{1}{\sqrt[n]{\omega(-n)}},\qquad R_+(\omega):=\lim_{n\to+\infty}\sqrt[n]{\omega(n)}$$

exist and $0<R_-(\omega)\leqslant 1\leqslant R_+(\omega)<\infty$. In the scalar case ($N=1$) the weighted Wiener algebra is commutative and the maximal ideal space of $W_\omega:=(W_\omega)_{1\times 1}$ is homeomorphic to the annulus

$$\Omega_\omega:=\big\{z\in\mathbb{C}\colon\ R_-(\omega)\leqslant|z|\leqslant R_+(\omega)\big\}.$$

The Gelfand transform of $a\in W_\omega$ is given by

$$\hat{a}(z)=\sum_{n=-\infty}^{\infty}a_nz^n\quad(z\in\Omega_\omega). \tag{11}$$

These results can be found in [12, Chapter III, Section 19.4] and in [14].

2.2. Canonical right and left Wiener–Hopf factorizations

Let I be the identity operator, P be the Riesz projection of $L^2 := L^2(\mathbb{T})$ onto H^2, $Q := I - P$, and define I, P, and Q on L^2_N elementwise. For $a \in L^\infty_{N\times N}$ and $t \in \mathbb{T}$, put $\tilde{a}(t) := a(1/t)$ and $(Ja)(t) := t^{-1}a(1/t)$. We consider *Toeplitz operators* $T(a) := PaP|\operatorname{Im} P, T(\tilde{a}) := JQaQJ|\operatorname{Im} P$ and *Hankel operators* $H(a) := PaQJ|\operatorname{Im} P$, $H(\tilde{a}) := JQaP|\operatorname{Im} P$. For a Banach algebra $\mathcal{A}$ we will denote by $G\mathcal{A}$ the group of all invertible elements of $\mathcal{A}$. Put

$$H^\infty_{N\times N} := \{a \in L^\infty_{N\times N}\colon a_{-n} = 0 \text{ for } n \in \mathbb{N}\},$$
$$\overline{H^\infty_{N\times N}} := \{a \in L^\infty_{N\times N}\colon a_n = 0 \text{ for } n \in \mathbb{N}\}.$$

Proposition 1. *Let $a \in L^\infty_{N\times N}$ admit two factorizations $a = u_-u_+ = v_+v_-$, where $u_+, v_+ \in GH^\infty_{N\times N}$ and $u_-, v_- \in G\overline{H^\infty_{N\times N}}$. Put*

$$b := v_-u_+^{-1}, \qquad c := u_-^{-1}v_+. \tag{12}$$

The operator $I - H(\tilde{c})H(b) = T(\tilde{c})T(\tilde{b})$ is invertible on H^2_N.

Proof. Widom [22] (see also [6, Proposition 2.7] and [7, Proposition 2.14]) formulated explicitly the very useful formula

$$T(fg) = T(f)T(g) + H(f)H(\tilde{g}) \quad \text{for } f, g \in L^\infty_{N\times N}. \tag{13}$$

Clearly, $cb = (u_-^{-1}v_+)(v_-u_+^{-1}) = u_-^{-1}au_+^{-1} = u_-^{-1}(u_-u_+)u_+^{-1} = E_N$, where E_N is the $N \times N$ identity matrix. From this and (13) it follows that

$$I - H(\tilde{c})H(b) = T(\tilde{c}\tilde{b}) - H(\tilde{c})H(b) = T(\tilde{c})T(\tilde{b}).$$

From (13), in view of $H(u_-^{-1}) = 0$ for $u_-^{-1} \in \overline{H^\infty_{N\times N}}$ and $H(\widetilde{v_+^{-1}}) = 0$ for $v_+^{-1} \in H^\infty_{N\times N}$, we obtain

$$\begin{aligned} T(\tilde{b})T\big(\widetilde{u_-^{-1}}\big)T(\widetilde{v_+}) &= T\big(\widetilde{v_-}\widetilde{u_+^{-1}}\widetilde{u_-^{-1}}\big)T(\widetilde{v_+}) = T\big(\widetilde{v_-}\widetilde{a^{-1}}\big)T(\widetilde{v_+}) \\ &= T\big(\widetilde{v_-}\widetilde{v_-^{-1}}\widetilde{v_+^{-1}}\big)T(\widetilde{v_+}) = T\big(\widetilde{v_+^{-1}}\big)T(\widetilde{v_+}) = T\big(\widetilde{v_+^{-1}}\widetilde{v_+}\big) = I. \end{aligned}$$

Analogously $T(\widetilde{u_-^{-1}})T(\widetilde{v_+})T(\tilde{b}) = I$. So $T^{-1}(\tilde{b}) = T(\widetilde{u_-^{-1}})T(\widetilde{v_+})$ and similarly $T^{-1}(\tilde{c}) = T(\widetilde{v_-})T(\widetilde{u_+^{-1}})$. Thus, $T(\tilde{c})T(\tilde{b})$ is invertible, too. □

For $\mathcal{A} \subset L^\infty$, put $\mathcal{A}^+_{N\times N} := \mathcal{A}_{N\times N} \cap H^\infty_{N\times N}$ and $\mathcal{A}^-_{N\times N} := \mathcal{A}_{N\times N} \cap \overline{H^\infty_{N\times N}}$.

Proposition 2. *Let ω be a weight satisfying* (6). *Every matrix function f in $G(W_\omega)_{N\times N}$ admits a right Wiener–Hopf factorization in $(W_\omega)_{N\times N}$, that is, there exist $f_\pm \in G(W_\omega)^\pm_{N\times N}$ such that $f(t) = f_-(t)\operatorname{diag}\{t^{\kappa_1}, \dots, t^{\kappa_N}\}f_+(t)$ for all $t \in \mathbb{T}$, where $\kappa_1, \dots, \kappa_N$ are some integers.*

Proof. Obviously, $W_\omega \subset L^\infty$, the set of all trigonometric polynomials is dense in W_ω, and $W_\omega^\pm \subset W_\omega$. So W_ω is a *decomposing algebra* in the sense of [7, Section 10.14]. Then the assertion follows from [7, Theorem 10.19]. $\square$

Proposition 3. *Let ω be a weight satisfying* (6) *and* (8). *If $a \in (W_\omega)_{N\times N}$ and the Toeplitz operators $T(a)$ and $T(\tilde{a})$ are invertible on H^2_N, then a admits canonical right and left Wiener–Hopf factorizations in $(W_\omega)_{N\times N}$, that is, there exist functions $u_-, v_- \in G(W_\omega)^-_{N\times N}$ and $u_+, v_+ \in G(W_\omega)^+_{N\times N}$ such that*

$$a(t) = u_-(t)u_+(t) = v_+(t)v_-(t) \quad \textit{for all } t \in \mathbb{T}. \tag{14}$$

Proof. It is very well known (see, e.g., [16, Sections 2.4–2.5]) that if $T(a)$ with $a \in W_{N\times N}$ is invertible on H^2_N, then a is invertible in $W_{N\times N}$,

$$\det a(t) \neq 0 \quad \text{for all } t \in \mathbb{T}, \tag{15}$$

and a admits a canonical right Wiener–Hopf factorization $a = a_-a_+$ in $W_{N\times N}$, that is $a_\pm \in GW_{N\times N}$. But, actually, a belongs to the smaller algebra $(W_\omega)_{N\times N}$. Under the condition (8), the maximal ideal space Ω_ω of the algebra W_ω coincides with $\mathbb{T}$. In this case, in view of (11), the Gelfand transform of $a \in W_\omega$ is simply given by $a(t)$ for $t \in \mathbb{T}$. Therefore, due to [14, Chapter XXX.8, Theorem 8.1], the condition (15) is equivalent to the invertibility of a in the algebra $(W_\omega)_{N\times N}$. By Proposition 2, a admits a right Wiener–Hopf factorization $a(t) = u_-(t)\operatorname{diag}\{t^{\kappa_1}, \dots, t^{\kappa_N}\}u_+(t)$ for all $t \in \mathbb{T}$ in the algebra $(W_\omega)_{N\times N}$, that is, $u_\pm \in G(W_\omega)_{N\times N}$. So, we have two factorizations $a = a_-a_+ = u_-\operatorname{diag}\{t^{\kappa_1}, \dots, t^{\kappa_N}\}u_+$ in the algebra $W_{N\times N}$. Since the partial indices $\kappa_1, \dots, \kappa_N$ are unique up to their order (see, e.g., [16, Theorem 1.2]), we conclude that $\kappa_1 = \dots = \kappa_N = 0$.

Let us prove that a admits also a canonical left factorization in the algebra $(W_\omega)_{N\times N}$. By [7, Proposition 7.19(b)], the invertibility of $T(\tilde{a})$ is equivalent to the invertibility of $T(a^{-1})$ (we have already known that $a^{-1} \in (W_\omega)_{N\times N}$). By what has just been proved, there exist $f_\pm \in G(W_\omega)^\pm_{N\times N}$ such that $a^{-1}(t) = f_-(t)f_+(t)$ for all $t \in \mathbb{T}$ and $f_\pm \in G(W_\omega)_{N\times N}$. Put $v_\pm = f_\pm^{-1}$. Then $v_\pm \in G(W_\omega)_{N\times N}$ and $a(t) = f_+^{-1}(t)f_-^{-1}(t) = v_+(t)v_-(t)$ for all $t \in \mathbb{T}$. $\square$

3. Schatten–von Neumann classes and operator determinants

3.1. Schatten–von Neumann classes

Let $\mathcal{H}$ be a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H}), \mathcal{C}_\infty(\mathcal{H})$, and $\mathcal{C}_0(\mathcal{H})$ the set of all bounded, compact, and finite rank operators on $\mathcal{H}$, respectively. Given an operator $A \in \mathcal{L}(\mathcal{H})$ define for $n \in \mathbb{Z}_+$,

$$s_n(A) := \inf\bigl\{\|A - F\|_{\mathcal{L}(\mathcal{H})}\colon\ F \in \mathcal{C}_0(\mathcal{H}),\ \dim F(\mathcal{H}) \leqslant n\bigr\}.$$

For $1 \leqslant p < \infty$, the collection of all operators $K \in \mathcal{L}(\mathcal{H})$ satisfying

$$\|K\|_p := \biggl(\sum_{n\geqslant 0} s_n^p(K)\biggr)^{1/p} < \infty$$

is denoted by $\mathcal{C}_p(\mathcal{H})$ and referred to as a *Schatten–von Neumann class*. Note that $\|K\|_\infty := \sup_{n\geqslant 0} s_n(K)$ is equal to the operator norm $\|K\|_{\mathcal{L}(\mathcal{H})}$.

Lemma 4. (See [17, Chapter III, Section 7.2]) *If $K_j \in \mathcal{C}_{p_j}(\mathcal{H})$ for $j = 1, \dots, m$ and $1/p_1 + 1/p_2 + \dots + 1/p_m \leqslant 1$, then $K = K_1 \dots K_m \in \mathcal{C}_p(\mathcal{H})$, where $1/p = 1/p_1 + 1/p_2 + \dots + 1/p_m$, and $\|K\|_p \leqslant \|K_1\|_{p_1} \|K_2\|_{p_2} \dots \|K_m\|_{p_m}$.*

Lemma 5. (See [22, Proposition 2.1]) *Let $B_n \in \mathcal{L}(\mathcal{H})$ converge strongly to $B \in \mathcal{L}(\mathcal{H})$ and let $C_n^* \in \mathcal{L}(\mathcal{H}^*)$ converge strongly to $C^* \in \mathcal{L}(\mathcal{H}^*)$. If $1 \leqslant p \leqslant \infty$ and $K \in \mathcal{C}_p(\mathcal{H})$, then $\|B_n K C_n - BKC\|_p \to 0$ as $n \to \infty$.*

We refer to [17] for more information about Schatten–von Neumann classes.

3.2. Trace

One can show that for every $K \in \mathcal{C}_1(\mathcal{H})$ and every orthonormal basis $\{e_n\}_{n\in\mathbb{Z}_+}$ of $\mathcal{H}$ the series $\sum_{n\in\mathbb{Z}_+} (Ke_n, e_n)$ is absolutely convergent and that its sum does not depend on a choice of a basis $\{e_n\}_{n\in\mathbb{Z}_+}$. This sum is denoted by $\operatorname{tr} K$ and referred to as the *trace* of $K \in \mathcal{C}_1(\mathcal{H})$. The operators in $\mathcal{C}_1(\mathcal{H})$ are called *trace class operators*. For every $A \in \mathcal{L}(\mathcal{H})$ and $K \in \mathcal{C}_1(\mathcal{H})$, we have $|\operatorname{tr} K| \leqslant \|K\|_1$ and $\operatorname{tr} AB = \operatorname{tr} BA$.

Proposition 6. *Let A_n and B_n be sequences of operators in $\mathcal{C}_0(\mathcal{H})$ such that $\|A_n\|_\infty = O(1)$ and $\|B_n\|_\infty = O(1)$ as $n \to \infty$. If $\|B_n\|_1 = o(1)$ or*

$$\|A_n B_n\|_1 = o(1), \quad \|B_n B_n\|_1 = o(1), \quad \operatorname{tr} B_n = o(1) \quad (n \to \infty), \tag{16}$$

then for $m \in \mathbb{N}$,

$$\sum_{j=1}^m \frac{1}{j} \operatorname{tr}\left[(A_n + B_n)^j\right] = \sum_{j=1}^m \frac{1}{j} \operatorname{tr} A_n^j + o(1) \quad (n \to \infty).$$

Proof. Since $\operatorname{tr} A_n B_n = \operatorname{tr} B_n A_n$, we have $\operatorname{tr}[A_n^\alpha B_n^\alpha] = \operatorname{tr}[B_n^\beta A_n^\alpha]$ for all $\alpha, \beta \in \mathbb{N}$. Using this property, one can get

$$\begin{aligned} S_n &:= \sum_{j=1}^m \frac{1}{j} \operatorname{tr}\left[(A_n + B_n)^j\right] - \sum_{j=1}^m \frac{1}{j} \operatorname{tr} A_n^j = \sum_{j=1}^m \sum_{k=0}^{j-1} \binom{j-1}{k} \operatorname{tr}\left[A_n^k B_n^{j-k}\right] \\ &= \operatorname{tr} B_n + \sum_{j=2}^m \sum_{k=0}^{j-1} \binom{j-1}{k} \operatorname{tr}\left[A_n^k B_n^{j-k}\right]. \end{aligned}$$

Then, by using $|\operatorname{tr} A| \leqslant \|A\|_1$, we get

$$|S_n| \leqslant |\operatorname{tr} B_n| + \sum_{j=2}^m \sum_{k=0}^{j-1} \binom{j-1}{k} \left\|A_n^k B_n^{j-k}\right\|_1. \tag{17}$$

If $\|B_n\|_1 = o(1)$, then $|\operatorname{tr} B_n| = o(1)$. By Lemma 4, we obtain from (17) that

$$|S_n| \leqslant \|B_n\|_1 + \sum_{j=2}^{m} \sum_{k=0}^{j-1} \binom{j-1}{k} \|A_n\|_\infty^k \|B_n\|_\infty^{j-k-1} \|B_n\|_1.$$

Taking into account that $\|A_n\|_\infty = O(1)$, $\|B_n\|_\infty = O(1)$ as $n \to \infty$, we get

$$|S_n| \leqslant \|B_n\|_1 + O\big(\|B_n\|_1\big) = o(1) \quad (n \to \infty).$$

If (16) holds, then from (17) and Lemma 4 we obtain

$$\begin{aligned} |S_n| &\leqslant |\operatorname{tr} B_n| \\ &\quad + \sum_{j=2}^{m} \left(\|B_n B_n\|_1 \|B_n\|_\infty^{j-2} + \sum_{k=1}^{j-2} \binom{j-1}{k} \|A_n B_n\|_1 \|A_n\|_\infty^{k-1} \|B_n\|_\infty^{j-k-1} \right) \\ &= |\operatorname{tr} B_n| + O\big(\|B_n B_n\|_1\big) + O\big(\|A_n B_n\|_1\big) = o(1) \quad (n \to \infty). \qquad \square \end{aligned}$$

3.3. *Operator determinants*

Let $A \in \mathcal{L}(\mathcal{H})$ be an operator of the form $I + K$ with $K \in \mathcal{C}_1(\mathcal{H})$. If $\{\lambda_j(K)\}_{j \geqslant 0}$ denotes the sequence of the nonzero eigenvalues of K (counted up to algebraic multiplicity), then $\sum_{j \geqslant 0} |\lambda_j(K)| < \infty$. Therefore the (possibly infinite) product $\prod_{j \geqslant 0}(1 + \lambda_j(K))$ is absolutely convergent. The determinant of A is defined by

$$\det A = \det(I + K) = \prod_{j \geqslant 0} \big(1 + \lambda_j(K)\big).$$

In the case where the spectrum of K consists only of 0 we put $\det(I + K) = 1$. The number $\det A$ is called the *determinant* of A. The next statement follows from the definition.

Lemma 7. *If P is a finite rank projection, then*

$$\det P(I + K)P = \det(I + PKP),$$

where the det *on the left refers to the ordinary finite-dimensional determinant for operators acting on* $\operatorname{Im} P$*, the image of* P*.*

3.4. *Regularized operator determinants*

Let $K \in \mathcal{C}_p(\mathcal{H})$, where $p > 1$ is an integer. A simple computation (see [20, Lemma 6.1]) shows that then

$$R_p(K) := (I + K) \exp\left\{ \sum_{j=1}^{p-1} \frac{(-K)^j}{j} \right\} - I \in \mathcal{C}_1(\mathcal{H}).$$

Thus, it is just natural to define $\det_p(I + K) := \det(I + R_p(K))$. One calls $\det_p(I + K)$ the *p-regularized determinant* of $I + K$.

Lemma 8. (See [20, Theorem 6.2]) *If $K \in \mathcal{C}_1(\mathcal{H})$, then*

$$\det{}_p(I+K) = \det(I+K)\exp\left\{\sum_{j=1}^{p-1}(-1)^j\frac{\operatorname{tr}(K^j)}{j}\right\}.$$

Lemma 9. (See [20, Theorem 6.5]) *Let $p \in \mathbb{N}$. If $\|K_n - K\|_p \to 0$, then $|\det{}_p(I+K_n) - \det{}_p(I+K)| \to 0$.*

Lemma 10. (See [20, Corollary 6.3]) *Let $p \in \mathbb{N}$ and $K \in \mathcal{C}_p(\mathcal{H})$. The operator $I+K$ is invertible on $\mathcal{H}$ if and only if $\det{}_p(I+K) \neq 0$.*

More information about operator determinants can be found in [15,17,20].

4. Asymptotic analysis

4.1. The Böttcher–Silbermann formula

Following [22] and [7, Sections 7.5–7.6], for $n \in \mathbb{Z}_+$ and $f \in L^\infty_{N\times N}$ define the operators P_n, W_n, and Q_n on H^2_N by

$$P_n : \sum_{k=0}^{\infty} f_k t^k \mapsto \sum_{k=0}^{n} f_k t^k, \qquad W_n : \sum_{k=0}^{\infty} f_k t^k \mapsto \sum_{k=0}^{n} f_{n-k} t^k, \qquad Q_n := I - P_n.$$

Let $a \in L^\infty_{N\times N}$. The operator $P_n T(a) P_n : P_n H^2_N \to P_n H^2_N$ may be identified with the finite block Toeplitz matrix $T_n(a) := [a_{j-k}]_{j,k=0}^n$. The following identities can be checked straightforwardly:

$$W_n^2 = P_n^2 = P_n, \qquad W_n P_n = P_n W_n = W_n, \qquad W_n T_n(a) W_n = T_n(\tilde{a}).$$

The following result is the starting point for our asymptotic analysis of block Toeplitz determinants. It was obtained by Böttcher and Silbermann in the late seventies [5] and is contained in [6, Section 6.15] and [7, Section 10.32].

Lemma 11. *Suppose $a \in L^\infty_{N\times N}$ satisfies the following assumptions*:

(i) *there are two factorizations $a = u_- u_+ = v_+ v_-$, where $u_+, v_+ \in GH^\infty_{N\times N}$ and $u_-, v_- \in G\overline{H^\infty_{N\times N}}$;*
(ii) *$u_- \in C_{N\times N}$ or $u_+ \in C_{N\times N}$.*

Then $D_n(a) \neq 0$ and

$$\frac{G(a)^{n+1}}{D_n(a)} = \det{}_1\left(I - \sum_{k=0}^{\infty} F_{n,k}\right)$$

for sufficiently large n, where $G(a)$ is given by (4)*, the series converges in the operator norm of H^2_N, and its terms $F_{n,k}$ are given for $n, k \in \mathbb{Z}_+$ by*

$$F_{n,0} := P_n T(c) Q_n T(b) P_n = W_n H(\tilde{c}) H(b) W_n, \tag{18}$$

$$F_{n,k} := P_n T(c) Q_n \big(Q_n H(b) H(\tilde{c}) Q_n\big)^k Q_n T(b) P_n \quad (k \geqslant 1) \tag{19}$$

with b, c *given by* (12)*.*

It is easy to see that the operator $\sum_{k=0}^{\infty} F_{n,k}$ is of finite rank, so the operator determinant in the above lemma is well defined.

4.2. *Estimates for norms of operator series*

Let $b, c \in L^\infty_{N\times N}$ and $F_{n,k}$ be given for $n, k \in \mathbb{Z}_+$ by (18)–(19). For $p \in \mathbb{N}$, put

$$A_n(p) := \sum_{k=0}^{p-1} F_{n,k}, \qquad B_n(p) := \sum_{k=p}^{\infty} F_{n,k}$$

where the latter series is understood as a formal expression.

Proposition 12. *Suppose* $p \in \mathbb{N}$ *and functions* $b, c \in L^\infty_{N\times N}$ *are such that* $H(b)H(\tilde{c})$ *belongs to* $\mathcal{C}_p(H^2_N)$*. Then*

$$\big\|B_n(1)\big\|_p = o(1), \quad \big\|B_n(p)\big\|_1 = o(1) \quad (n \to \infty). \tag{20}$$

Proof. By Lemma 5, because $Q_n \to 0$ strongly and $H(b)H(\tilde{c}) \in \mathcal{C}_p(H^2_N)$, we have

$$\lim_{n\to\infty} \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p = 0. \tag{21}$$

Hence, there is a number $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$,

$$\big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_\infty \leqslant \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p \leqslant 1/2. \tag{22}$$

For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, by Lemma 4,

$$\begin{aligned} \|F_{n,k}\|_p &\leqslant \big\|P_n T(c) Q_n\big\|_\infty \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p \\ &\quad \times \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_\infty^{k-1} \big\|Q_n T(b) P_n\big\|_\infty \\ &\leqslant C(b,c) \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_\infty^{k-1}, \end{aligned} \tag{23}$$

with $C(b,c) := \|T(c)\|_{\mathcal{L}(H^2_N)} \|T(b)\|_{\mathcal{L}(H^2_N)}$. From (23) and (22) it follows that

$$\begin{aligned} \big\|B_n(1)\big\|_p &\leqslant C(b,c) \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p \sum_{k=1}^{\infty} \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_\infty^{k-1} \\ &\leqslant C(b,c) \big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p \sum_{k=1}^{\infty} 2^{1-k} \\ &= O\big(\big\|Q_n H(b) H(\tilde{c}) Q_n\big\|_p\big). \end{aligned} \tag{24}$$

Similarly, by Lemma 4, for $k \geqslant p$ and $n \in \mathbb{Z}_+$,

$$\|F_{n,k}\|_1 \leqslant C(b,c)\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^p \big\|Q_n H(b)H(\tilde{c})Q_n\big\|_\infty^{k-p}. \tag{25}$$

From (25) and (22) it follows that for $n \geqslant n_0$,

$$\begin{aligned}\big\|B_n(p)\big\|_1 &\leqslant C(b,c)\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^p \sum_{k=p}^{\infty}\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_\infty^{k-p}\\ &= O\big(\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^p\big).\end{aligned} \tag{26}$$

Combining (24) and (26) with (21), we immediately get (20). □

Proposition 13. *Suppose $p \in \mathbb{N} \setminus \{1\}$ and functions $b, c \in L^\infty_{N\times N}$ are such that $H(\tilde{c})H(b)$ and $H(b)H(\tilde{c})$ belong to $\mathcal{C}_p(H^2_N)$. Then*

$$\|F_{n,p-1}F_{n,p-1}\|_1 = o(1), \quad \big\|A_n(p-1)F_{n,p-1}\big\|_1 = o(1) \quad (n \to \infty).$$

Proof. Since $H(\tilde{c})H(b)$ and $H(b)H(\tilde{c})$ belong to $\mathcal{C}_p(H^2_N)$, the operators $F_{n,j}F_{n,p-1}$, where $j \in \{0,\dots,p-1\}$, contain at least p terms in $\mathcal{C}_p(H^2_N)$. Therefore, by Lemma 4, we can estimate the 1-norms of $F_{n,j}F_{n,p-1}$ by products of p p-norms of those operators in $\mathcal{C}_p(H^2_N)$ and ∞-norms of the rest operators. More precisely, be Lemma 4,

$$\begin{aligned}\|F_{n,0}F_{n,p-1}\|_1 &\leqslant \|W_n\|_\infty \big\|H(\tilde{c})H(b)\big\|_p \|W_n\|_\infty\\ &\quad\times \big\|P_n T(c)Q_n\big\|_\infty \big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^{p-1}\big\|Q_n T(b)P_n\big\|_\infty\\ &= O\big(\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^{p-1}\big).\end{aligned} \tag{27}$$

Similarly, for $j \in \{1,\dots,p-1\}$,

$$\|F_{n,j}F_{n,p-1}\|_1 \leqslant C_j(b,c)\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^p, \tag{28}$$

where $C_j(b,c) := (\|T(b)\|_{\mathcal{L}(H^2_N)}\|T(c)\|_{\mathcal{L}(H^2_N)})^2\|H(b)H(\tilde{c})\|^{j-1}_{\mathcal{L}(H^2_N)}$. From (27) and (28) it follows that

$$\big\|A_n(p-1)F_{n,p-1}\big\|_1 = O\big(\big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^{p-1} + \big\|Q_n H(b)H(\tilde{c})Q_n\big\|_p^p\big).$$

From this equality and (21) we get $\|A_n(p-1)F_{n,p-1}\|_1 = o(1)$ as $n \to \infty$. By (28) and (21), $\|F_{n,p-1}F_{n,p-1}\|_1 = o(1)$ as $n \to \infty$. □

The trick of the estimate of the 1-norm of $F_{n,j}F_{n,p-1}$ with $j \in \{0,\dots,p-1\}$ via exactly p p-norms and the rest ∞-norms is crucial in the above proof. It was communicated to the author by Albrecht Böttcher [4].

Proposition 14. *If $p \in \mathbb{N}$ and $b, c \in L^\infty_{N\times N}$ are such that $H(b)H(\tilde{c})$ is compact on H^2_N, then $\|A_n(p)\|_\infty = O(1)$ and $\|B_n(p)\|_\infty = O(1)$ as $n \to \infty$.*

This proposition is proved by analogy with Propositions 12 and 13.

4.3. Higher order asymptotic formulas for Toeplitz determinants

The following theorem is implicitly contained in [5, Theorem 8], [7, Theorem 10.35], and especially [6, Section 6.23]. This theorem is “smoothness free” in the sense that we assume only that $H(\tilde{c})H(b)$ and $H(b)H(\tilde{c})$ belong to the Schatten–von Neumann class $\mathcal{C}_p(H^2_N)$. Notice also that necessary and sufficient conditions guaranteeing that the product of two Hankel operators belongs to $\mathcal{C}_p(H^2)$ are unknown (see [7, Sections 4.50 and 10.56]).

Theorem 15. *Let $a \in L^\infty_{N\times N}$ satisfy the conditions* (i) *and* (ii) *of Lemma* 11. *Define the constant $G(a)$, the functions b, c, and the operators $F_{n,k}$ by* (4), (12), *and* (18)–(19), *respectively. If $p \in \mathbb{N}$ and $H(\tilde{c})H(b)$, $H(b)H(\tilde{c})$ belong to $\mathcal{C}_p(H^2_N)$, then*

$$\lim_{n\to\infty} \frac{D_n(a)}{G(a)^{n+1}} \exp\left\{-\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\left[\left(\sum_{k=0}^{p-1} F_{n,k}\right)^j\right]\right\} = \frac{1}{\det_p T(\tilde{c})T(\tilde{b})}. \tag{29}$$

Proof. Let us prove that the formula holds for $p = 1$, that is,

$$\lim_{n\to\infty} \frac{D_n(a)}{G(a)^{n+1}} = \frac{1}{\det_1 T(\tilde{c})T(\tilde{b})}. \tag{30}$$

From Lemma 11 it follows that for sufficiently large n,

$$\frac{G(a)^{n+1}}{D_n(a)} = \det_1\left(I - \sum_{k=0}^{\infty} F_{n,k}\right). \tag{31}$$

From Proposition 12 and Lemma 9 we get

$$\det_1\left(I - \sum_{k=0}^{\infty} F_{n,k}\right) = \det_1(I - F_{n,0}) + o(1). \tag{32}$$

From Lemma 7, taking into account that $P_n^2 = P_n$ and

$$F_{n,0} = W_n H(\tilde{c})H(b)W_n = P_n T(c) Q_n T(b) P_n \tag{33}$$

we obtain $\det_1(I - F_{n,0}) = \det_1 P_n(I - F_{n,0})P_n$. From the latter equality and the obvious identity $\det_1 P_n A P_n = \det_1 W_n A W_n$ it follows that

$$\det_1(I - F_{n,0}) = \det_1 W_n(I - F_{n,0})W_n. \tag{34}$$

On the other hand, since $W_n^2 = P_n = P_n^2$, we have

$$W_n(I - F_{n,0})W_n = P_n\bigl(I - P_n H(\tilde{c})H(b)P_n\bigr)P_n. \tag{35}$$

Combining (34), (35), and Lemma 7, we obtain

$$\det_1(I - F_{n,0}) = \det_1\bigl(I - P_n H(\tilde{c})H(b)P_n\bigr). \tag{36}$$

Since $H(\tilde{c})H(b) \in \mathcal{C}_1(H^2_N)$ and $P_n \to I$ strongly, by Lemma 5,

$$\lim_{n\to\infty} \bigl\| P_n H(\tilde{c})H(b)P_n - H(\tilde{c})H(b)\bigr\|_1 = 0. \tag{37}$$

From (37) and Lemma 9 it follows that

$$\det{}_1\big(I - P_n H(\tilde{c})H(b)P_n\big) = \det{}_1\big(I - H(\tilde{c})H(b)\big) + o(1) \quad (n \to \infty). \tag{38}$$

Combining (31), (32), (36), and (38), we arrive at

$$\frac{G(a)^{n+1}}{D_n(a)} = \det{}_1(I - F_{n,0}) + o(1) = \det{}_1\big(I - H(\tilde{c})H(b)\big) + o(1). \tag{39}$$

By Proposition 1, the operator $T(\tilde{c})T(\tilde{b}) = I - H(\tilde{c})H(b)$ is invertible on H^2_N. From (39) and Lemma 10 we conclude that (30) holds.

Let $p \in \mathbb{N} \setminus \{1\}$. It is clear that $\sum_{k=0}^{\infty} F_{n,k} = \sum_{k=0}^{p-1} F_{n,k} + \sum_{k=p}^{\infty} F_{n,k} =: A_n(p) + B_n(p)$ is of finite rank. Therefore we can apply Lemma 8 and get

$$\det{}_p\left(I - \sum_{k=0}^{\infty} F_{n,k}\right) = \det{}_1\big(I - A_n(p) - B_n(p)\big) \times \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\big[\big(A_n(p) + B_n(p)\big)^j\big]\right\}. \tag{40}$$

From (31) and (40) it follows that

$$\frac{G(a)^{n+1}}{D_n(a)} \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\big[\big(A_n(p) + B_n(p)\big)^j\big]\right\} = \det{}_p\left(I - \sum_{k=0}^{\infty} F_{n,k}\right). \tag{41}$$

From Proposition 12 and Lemma 9 we get

$$\det{}_p\left(I - \sum_{k=0}^{\infty} F_{n,k}\right) = \det{}_p(I - F_{n,0}) + o(1). \tag{42}$$

By Propositions 14 and 12, $\|A_n(p)\|_\infty = O(1)$, $\|B_n(p)\|_\infty = O(1)$, and $\|B_n(p)\|_1 = o(1)$. Then applying Proposition 6 with $A_n = A_n(p)$, $B_n = B_n(p)$, and $m = p - 1$, we get

$$\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\big[\big(A_n(p) + B_n(p)\big)^j\big] = \sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\big[A_n^j(p)\big] + o(1) \quad (n \to \infty). \tag{43}$$

Combining (41)–(43), we arrive at

$$\frac{G(a)^{n+1}}{D_n(a)} \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\big[A_n^j(p)\big]\right\} = \det{}_p(I - F_{n,0}) + o(1). \tag{44}$$

Since $F_{n,0}$ is a finite-rank operator, it belongs to $\mathcal{C}_1(H^2_N)$. Then, by Lemma 8,

$$\det{}_p(I - F_{n,0}) = \det{}_1(I - F_{n,0}) \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\big[F_{n,0}^j\big]\right\}. \tag{45}$$

From (33) and $W_n^2 = P_n$ it follows that for $j \in \{1, \dots, p-1\}$,

$$F_{n,0}^j = W_n H(\tilde{c})H(b)P_n H(\tilde{c})H(b)P_n \times \cdots \times P_n H(\tilde{c})H(b)W_n.$$

From the latter equality and the obvious identity $\operatorname{tr} W_n A W_n = \operatorname{tr} P_n A P_n$ it follows that for $j \in \{1, \dots, p-1\}$,

$$\operatorname{tr}\bigl[F_{n,0}^j\bigr] = \operatorname{tr}\bigl[\bigl(P_n H(\tilde{c}) H(b) P_n\bigr)^j\bigr]. \tag{46}$$

From (45), (36), (46), and Lemma 8 it follows that

$$\begin{aligned}\det{}_p(I - F_{n,0}) &= \det{}_1\bigl(I - P_n H(\tilde{c}) H(b) P_n\bigr) \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\bigl[\bigl(P_n H(\tilde{c}) H(b) P_n\bigr)^j\bigr]\right\} \\ &= \det{}_p\bigl(I - P_n H(\tilde{c}) H(b) P_n\bigr). \end{aligned}\tag{47}$$

Since $H(\tilde{c}) H(b) \in \mathcal{C}_p(H_N^2)$ and $P_n \to I$ strongly, by Lemma 5,

$$\lim_{n\to\infty} \bigl\| P_n H(\tilde{c}) H(b) P_n - H(\tilde{c}) H(b) \bigr\|_p = 0. \tag{48}$$

From (48) and Lemma 9 we obtain

$$\det{}_p\bigl(I - P_n H(\tilde{c}) H(b) P_n\bigr) = \det{}_p\bigl(I - H(\tilde{c}) H(b)\bigr) + o(1) \quad (n \to \infty). \tag{49}$$

Combining (44), (47), and (49), we arrive at

$$\frac{G(a)^{n+1}}{D_n(a)} \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j} \operatorname{tr}\left[\left(\sum_{k=0}^{p-1} F_{n,k}\right)^j\right]\right\} = \det{}_p\bigl(I - H(\tilde{c}) H(b)\bigr) + o(1). \tag{50}$$

By Proposition 1, the operator $T(\tilde{c}) T(\tilde{b}) = I - H(\tilde{c}) H(b)$ is invertible on H_N^2. From (50) and Lemma 10 we get (29) for $p \in \mathbb{N} \setminus \{1\}$. □

5. Refined asymptotic formulas for weighted Wiener algebras

In this section we will show that under natural conditions on the weight $\omega : \mathbb{Z} \to [1, \infty)$, the invertibility of $T(a)$ and $T(\tilde{a})$ with $a \in (W_\omega)_{N\times N}$ implies the existence of canonical factorizations $a = u_- u_+ = v_- v_+$ such that $H(\tilde{c}) H(b)$ and $H(b) H(\tilde{c})$ belong to $\mathcal{C}_p(H_N^2)$, so all the conditions of Theorem 15 are fulfilled. Moreover, we will show how to remove the term $F_{n,p-1}$ on the left of (29) under one more condition on the weight.

5.1. *Estimates for norms of truncated Hankel and Toeplitz operators*

Now we prove estimates for norms of truncated Hankel and Toeplitz operators with symbols in weighted Wiener algebras.

Proposition 16. *Let ω be a weight satisfying* (6)–(7). *If $b, c \in (W_\omega)_{N\times N}$, then for every $n \in \mathbb{Z}_+$,*

$$\bigl\| Q_n H(b) \bigr\|_{\mathcal{L}(H_N^2)} \leqslant \frac{1}{\omega(n+1)} \sum_{k=n+1}^{\infty} \|b_k\| \omega(k), \tag{51}$$

$$\bigl\| H(\tilde{c}) Q_n \bigr\|_{\mathcal{L}(H_N^2)} \leqslant \frac{1}{\omega(-(n+1))} \sum_{k=n+1}^{\infty} \|c_{-k}\| \omega(-k). \tag{52}$$

Proof. It is not difficult to show that

$$Q_nH(b) = Q_nH(Pb - P_nPb), \qquad H(\tilde{c})Q_n = H\big((Qc)\tilde{} - P_n(Qc)\tilde{}\big)Q_n. \tag{53}$$

It is well known (and not difficult to prove) that $\|H(f)\|_{\mathcal{L}(H^2_N)} \leqslant \|f\|_{1,N}$ for $f \in W_{N\times N}$. Applying this fact to (53), we get

$$\big\|Q_nH(b)\big\|_{\mathcal{L}(H^2_N)} \leqslant \|Pb - P_nPb\|_{1,N} = \sum_{k=n+1}^{\infty} \|b_k\|, \tag{54}$$

$$\big\|H(\tilde{c})Q_n\big\|_{\mathcal{L}(H^2_N)} \leqslant \big\|(Qc)\tilde{} - P_n(Qc)\tilde{}\big\|_{1,N} = \sum_{k=n+1}^{\infty} \|c_{-k}\|. \tag{55}$$

Taking into account that $\omega(\pm k) \leqslant \omega(\pm(k+1))$ for $k \in \mathbb{Z}_+$, from (54) and (55) we obtain (51) and (52), respectively (notice that the series in (51) and (52) are convergent since $b, c \in (W_\omega)_{N\times N}$). □

Put $\Delta_0 := P_0$ and $\Delta_j := P_j - P_{j-1}$ for $j \in \{1, \dots, n\}$.

Proposition 17. *Let ω be a weight satisfying* (6)–(7). *If $b, c \in (W_\omega)_{N\times N}$, then for every $n \in \mathbb{Z}_+$ and every $j \in \{0, \dots, n\}$,*

$$\big\|Q_nT(b)\Delta_j\big\|_{\mathcal{L}(H^2_N)} \leqslant \frac{1}{\omega(n-j+1)} \sum_{k=n-j+1}^{\infty} \|b_k\|\omega(k),$$

$$\big\|\Delta_jT(c)Q_n\big\|_{\mathcal{L}(H^2_N)} \leqslant \frac{1}{\omega(-(n-j+1))} \sum_{k=n-j+1}^{\infty} \|c_{-k}\|\omega(-k).$$

Proof. Manipulating with Fourier coefficients of the functions $Q_nT(b)\Delta_j\varphi$ and $\Delta_jT(c)Q_n\varphi$, it is not difficult to show that

$$\big\|Q_nT(b)\Delta_j\big\|_{\mathcal{L}(H^2_N)} \leqslant \sum_{k=n-j+1}^{\infty} \|b_k\|, \qquad \big\|\Delta_jT(c)Q_n\big\|_{\mathcal{L}(H^2_N)} \leqslant \sum_{k=n-j+1}^{\infty} \|c_{-k}\|.$$

From these inequalities and (7) we immediately get the assertion. □

5.2. *Sufficient conditions guaranteeing elimination of $F_{n,p-1}$*

Using the results of the preceding section, we prove now that if b and c are sufficiently smooth, then the conditions of Proposition 6 are fulfilled with $m = p-1$, $A_n = \sum_{k=0}^{p-2} F_{n,k}$, and $B_n = F_{n,p-1}$ For a weight $\omega : \mathbb{Z} \to [1, \infty)$, put

$$\varphi_k := \frac{1}{\omega(k+1)\omega(-k-1)} \quad (k \in \mathbb{Z}_+). \tag{56}$$

Lemma 18. *Suppose $p \in [1, \infty)$ and ω is a weight satisfying* (6)–(7). *Assume $b, c \in (W_\omega)_{N\times N}$ and $\sum_{k=0}^{\infty} \varphi_k^p < \infty$. Then $H(\tilde{c})H(b)$, $H(b)H(\tilde{c}) \in \mathcal{C}_p(H^2_N)$.*

Proof. This lemma is proved by a repetition with minor changes of the proof of [6, Lemma 6.19(i)] (see also [7, Lemma 10.34(i)]).

Proposition 19. *Suppose* $p \in \mathbb{N} \setminus \{1\}$ *and* ω *is a weight satisfying* (6)–(7). *If* $b, c \in (W_\omega)_{N\times N}$ *and*

$$\lim_{n\to\infty} \varphi_n^{p-1}\left(\sum_{j=0}^{n} \varphi_j\right) = 0, \tag{57}$$

then $\operatorname{tr} F_{n,p-1} = o(1)$ *as* $n \to \infty$.

Proof. If $A \in \mathcal{L}(H_N^2)$, then

$$\operatorname{tr} P_n A P_n = \sum_{j=0}^{n} \operatorname{tr} \Delta_j A \Delta_j. \tag{58}$$

Each of the operators $\Delta_j A \Delta_j$ is of rank one. For a rank one operator its 1-norm coincides with its ∞-norm. Therefore from (58) it follows that

$$|\operatorname{tr} P_n A P_n| \leqslant \sum_{j=0}^{n} |\operatorname{tr} \Delta_j A \Delta_j| \leqslant \sum_{j=0}^{n} \|\Delta_j A \Delta_j\|_1 = \sum_{j=0}^{n} \|\Delta_j A \Delta_j\|_{\mathcal{L}(H_N^2)}. \tag{59}$$

Let $A = F_{n,p-1}$. Taking into account that $\Delta_j P_n = P_n \Delta_j$ for $j \in \{0, \dots, n\}$, we obtain for $n \in \mathbb{Z}_+$,

$$\begin{aligned} \|\Delta_j F_{n,p-1} \Delta_j\|_{\mathcal{L}(H_N^2)} &\leqslant \big\|\Delta_j T(c) Q_n\big\|_{\mathcal{L}(H_N^2)} \big\|Q_n H(b)\big\|_{\mathcal{L}(H_N^2)}^{p-1} \\ &\quad \times \big\|H(\tilde{c}) Q_n\big\|_{\mathcal{L}(H_N^2)}^{p-1} \big\|Q_n T(b) \Delta_j\big\|_{\mathcal{L}(H_N^2)}. \end{aligned} \tag{60}$$

Since $b, c \in (W_\omega)_{N\times N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$,

$$\sum_{k=n+1}^{\infty} \|b_k\| \omega(k) \leqslant 1, \qquad \sum_{k=n+1}^{\infty} \|c_{-k}\| \omega(-k) \leqslant 1. \tag{61}$$

From (60)–(61) and Propositions 16–17 it follows that for $n \geqslant n_0$,

$$\|\Delta_j F_{n,p-1} \Delta_j\|_{\mathcal{L}(H_N^2)} \leqslant \varphi_{n-j} \varphi_n^{p-1}, \quad j \in \{0, \dots, n\}. \tag{62}$$

From (59) and (62) we obtain for $n \geqslant n_0$,

$$|\operatorname{tr} F_{n,p-1}| = |\operatorname{tr} P_n F_{n,p-1} P_n| \leqslant \varphi_n^{p-1}\left(\sum_{j=0}^{n} \varphi_j\right). \tag{63}$$

From (57) and (63) we immediately get $\operatorname{tr} F_{n,p-1} = o(1)$ as $n \to \infty$. $\square$

5.3. The case of general weights

Theorem 20. *Let ω be a weight satisfying* (6)–(8) *and φ_k be given for $k \in \mathbb{Z}_+$ by* (56)*. Suppose $a \in (W_\omega)_{N\times N}$ and $T(a), T(\tilde{a})$ are invertible on H^2_N. Define the constant $G(a)$ by*

$$G(a) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\det a\big(e^{i\theta}\big)\,d\theta\right\} \tag{64}$$

and define the functions b, c, and the operators $F_{n,k}$ by (12)*, and* (18)–(19)*, respectively.*

(a) *If $\sum_{k=0}^{\infty}\varphi_k < \infty$, then*

$$H(\tilde{c})H(b) = I - T(\tilde{c})T(\tilde{b}) \in \mathcal{C}_1\big(H^2_N\big), \tag{65}$$

$$H(a)H\big(\tilde{a}^{-1}\big) = I - T(a)T\big(a^{-1}\big) \in \mathcal{C}_1\big(H^2_N\big), \tag{66}$$

$$\lim_{n\to\infty}\frac{D_n(a)}{G(a)^{n+1}} = \frac{1}{\det{}_1 T(\tilde{c})T(\tilde{b})} = \det{}_1 T(a)T\big(a^{-1}\big). \tag{67}$$

(b) *If $p \in \mathbb{N}\setminus\{1\}$ and $\sum_{k=0}^{\infty}\varphi_k^p < \infty$, then*

$$H(\tilde{c})H(b) = I - T(\tilde{c})T(\tilde{b}) \in \mathcal{C}_p\big(H^2_N\big) \tag{68}$$

and (29) *holds.*

(c) *Suppose the conditions of* (b) *are fulfilled and*

$$\lim_{n\to\infty}\varphi_n^{p-1}\left(\sum_{j=0}^{n}\varphi_j\right) = 0,$$

then (68) *holds and one can remove $F_{n,p-1}$ in* (29)*, that is,*

$$\lim_{n\to\infty}\frac{D_n(a)}{G(a)^{n+1}}\exp\left\{-\sum_{j=1}^{p-1}\frac{1}{j}\operatorname{tr}\left[\left(\sum_{k=0}^{p-2}F_{n,k}\right)^j\right]\right\} = \frac{1}{\det{}_p T(\tilde{c})T(\tilde{b})}. \tag{69}$$

Proof. Let $p \in \mathbb{N}$. By Proposition 3, the function a admits canonical left and right factorizations (14) in the algebra $(W_\omega)_{N\times N}$. Therefore, the functions b and c are correctly defined by (12) and $b, c \in (W_\omega)_{N\times N}$. Lemma 18 gives that $H(\tilde{c})H(b)$ and $H(b)H(\tilde{c})$ belong to $\mathcal{C}_p(H^2_N)$.

(a) Let $p = 1$. Since $a, a^{-1} \in (W_\omega)_{N\times N}$, by Lemma 18, $H(a)H(\tilde{a}^{-1}) \in \mathcal{C}_1(H^2_N)$. In view of (13), $I - H(a)H(\tilde{a}^{-1}) = T(a)T(a^{-1})$. Due to the corollary from [6, Lemma 6.8],

$$\lim_{n\to\infty}\frac{D_n(a)}{G(a)^{n+1}} = \det{}_1 T(a)T\big(a^{-1}\big).$$

The rest follows from Theorem 15. Part (a) is proved.

(b) This immediately follows from Theorem 15.

(c) It is sufficient to show that

$$\sum_{j=1}^{p-1}\frac{1}{j}\operatorname{tr}\left[\left(\sum_{k=0}^{p-1}F_{n,k}\right)^j\right] = \sum_{j=1}^{p-1}\frac{1}{j}\operatorname{tr}\left[\left(\sum_{k=0}^{p-2}F_{n,k}\right)^j\right] + o(1) \quad (n\to\infty). \tag{70}$$

Let $A_n := A_n(p-1) = \sum_{k=0}^{p-2} F_{n,k}$, $B_n = F_{n,p-1}$. By Proposition 14, $\|A_n\|_\infty = O(1)$ and it is easy to see that $\|B_n\|_\infty = O(1)$. In view of Proposition 13, $\|A_n B_n\|_1 = o(1)$ and $\|B_n B_n\|_1 = o(1)$. Applying Proposition 6 with A_n, B_n and $m = p-1$, we get (70). Then (69) follows from (29) and (70). □

5.4. *The case of power weights*

For $\alpha, \beta > 0$ denote by $W^{\alpha,\beta}$ the weighted Wiener algebra W_ϱ with the power weight ϱ given by (5).

The next theorem was obtained in the late seventies by Böttcher and Silbermann [5, Theorem 8] and it is contained in their books [6, Theorem 6.20] and [7, Theorem 10.35]. It is easy to see that it follows from Theorem 20(a), (c).

Theorem 21. *Suppose $a \in W^{\alpha,\beta}_{N\times N}$ and $T(a), T(\tilde{a})$ are invertible on H^2_N. Define the constant $G(a)$, the functions b, c, and the operators $F_{n,k}$ by* (64), (12), *and* (18)–(19), *respectively.*

(a) *If $\alpha + \beta > 1$, then* (65)–(67) *hold.*
(b) *If $p \in \mathbb{N} \setminus \{1\}$ and $\alpha + \beta > 1/p$, then* (68)–(69) *hold.*

5.5. *Example*

In this section we will show that Theorem 20 is stronger than Theorem 21.

Proposition 22. *Suppose $\omega : \mathbb{Z} \to [1, \infty)$ is a weight such that $\omega(0) = 1$,* (7) *holds, and there exist constants $C_\pm > 0$ such that $\omega(\pm 2n) \leqslant C_\pm \omega(\pm n)$ for $n \in \mathbb{Z}_+$. Then* (8) *holds and $\omega(i+j) \leqslant C\omega(i)\omega(j)$ for all $i, j \in \mathbb{Z}$ with $C := \max\{C_-, C_+\}$.*

The proof of this statement is straightforward and it is omitted.

Theorem 23. *There exist a weight $\omega : \mathbb{Z} \to [1, \infty)$ satisfying* (6)–(8) *and*

$$\sum_{k=1}^{\infty} \bigl(\omega(k)\omega(-k)\bigr)^{-2} < \infty, \qquad \lim_{n\to\infty} \frac{1}{\omega(n)\omega(-n)} \left(\sum_{j=1}^{n} \frac{1}{\omega(j)\omega(-j)} \right) = 0,$$

and a function $a \in W_\omega$ such that $a \notin W^{\alpha,\beta}$ for all $\alpha + \beta > 1/2$, $a(t) \neq 0$ for all $t \in \mathbb{T}$, and the winding number of $a(t)$ about the origin is equal to zero. That is, Theorem 20(c) *is applicable to the function a with $p = 2$, but Theorem* 21(b) *is not.*

Proof. Let

$$\tilde{\omega}(k) := \Bigl(\sqrt{|k|+1}\,\log\bigl(|k|+e\bigr)\Bigr)^{1/2} \quad (k \in \mathbb{Z}).$$

Obviously, $\tilde{\omega}(0) = 1$, $\tilde{\omega}(\pm n) \leqslant \tilde{\omega}(\pm(n+1))$ and $\tilde{\omega}(\pm 2n) \leqslant (2\sqrt{2})^{1/2}\tilde{\omega}(\pm n)$ for $n \in \mathbb{Z}_+$. By Proposition 22, the weight $\omega(k) := (2\sqrt{2})^{1/2}\tilde{\omega}(k) =: C\tilde{\omega}(k)$, where $k \in \mathbb{Z}$, satisfies (6)–(8). Clearly,

$$\sum_{k=1}^{\infty}\bigl(\omega(k)\omega(-k)\bigr)^{-2}=C^{-4}\sum_{k=1}^{\infty}\frac{1}{(k+1)\log^2(k+e)}<\infty,$$

$$\begin{aligned}\frac{1}{\omega(n)\omega(-n)}&\sum_{j=1}^{n}\frac{1}{\omega(j)\omega(-j)}\\ &\leqslant\frac{C^{-4}}{\sqrt{n}\log(n+e)}\sum_{j=1}^{n}\frac{1}{\sqrt{n}}=\frac{1}{\sqrt{n}\log(n+e)}O\bigl(n^{1-1/2}\bigr)\\ &=O\left(\frac{1}{\log(n+e)}\right)=o(1)\quad\text{as } n\to\infty.\end{aligned}$$

Let the function $a:\mathbb{T}\to\mathbb{C}$ be given by

$$a(t):=\alpha_0+\sum_{k=1}^{\infty}\alpha_k\bigl(t^k+t^{-k}\bigr),\qquad \alpha_k=\frac{1}{(k+1)^{5/4}\log^2(k+e)}\qquad(k\in\mathbb{N})$$

and α_0 will be chosen later. It is easy to see that

$$\sum_{k=1}^{\infty}\alpha_k\tilde{\omega}(-k)+\sum_{k=1}^{\infty}\alpha_k\tilde{\omega}(k)=2\sum_{k=1}^{\infty}\frac{1}{(k+1)\log^{3/2}(k+e)}<\infty,$$

whence $a\in W_\omega$.

Suppose $\alpha+\beta>1/2$. If $a\in W^{\alpha,\beta}$, then the series

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)^{5/4-\alpha}\log^2(k+e)}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^{5/4-\beta}\log^2(k+e)}$$

is convergent. But this is possibly only when $5/4-\alpha\geqslant 1$ and $5/4-\beta\geqslant 1$. Therefore, $\alpha+\beta\leqslant 1/2$ and we get a contradiction. Thus, $a\notin W^{\alpha,\beta}$ for $\alpha+\beta>1/2$. That is, it is not possible to apply Theorem 21(b) to a with $p=2$.

Finally, let us choose α_0 so that $a(t)\neq 0$ and the winding number of a about the origin is zero. These conditions are equivalent to the invertibility of both $T(a)$ and $T(\tilde{a})$ on H^2 (in the scalar case!) (see [7, Theorem 2.42(c) and Proposition 7.19(c)]). So, all the hypotheses of Theorem 20(c) are fulfilled for a and $p=2$. □

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